Recollements and Hochschild theory

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Abstract

Two constructions of recollements of derived categories of algebras are provided. Triangulated functors in recollements of derived categories of algebras and tensor product algebras are realized as derived functors of the same forms. These results are applied to observe the relations between recollements of derived categories of algebras and smoothness and Hochschild (co)homology of algebras.

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1 Introduction

Recollements of triangulated categories are "short exact sequences" of triangulated categories, which were introduced by Beilinson-Bernstein-Deligne [5] and play an important role in algebraic geometry [5], representation theory [10, 35], etc. Let k be a field and $\otimes := \otimes_k$. Throughout the paper, all algebras are assumed to be associative k-algebras with identity, and all modules are right unitary modules unless stated otherwise. Here, we focus on recollements of derived categories of algebras, i.e., all triangulated categories in the recollements are derived categories of algebras, which are closed related to tilting theory [1, 23, 32], (co)localization theory [31, 28], some important homological invariants of algebras such as global dimension [41, 22, 2], finitistic dimension [17], Hochschild homology and cyclic homology [22], and so on.

In this paper, on one hand, in the interior of recollement theory, we shall provide two constructions of recollements of derived categories of algebras

and realize triangulated functors in recollements of derived categories of algebras and tensor product algebras as derived functors of the same forms in the spirit of [37]. On the other hand, as applications, we shall observe the relations between recollements of derived categories of algebras and smoothness, i.e., finiteness of Hochschild dimension, and Hochschild (co)homology of algebras. Note that the relations between (perfect) recollements of derived categories of algebras and Hochschild homology even cyclic homology had been clarified already in [22]. The paper is organized as follows: In section 2, we shall introduce perfect recollements of derived categories of algebras which correspond to "derived triangular matrix (differential graded) algebras", and give a criterion for the derived category of an algebra to admit a perfect recollement of derived categories of algebras. In section 3, we shall construct (perfect) recollements of derived categories of tensor product algebras and opposite algebras respectively from a (perfect) recollement of derived categories of algebras. Applying the constructions provided in section 3, we shall show in section 4 that, in a perfect recollement of derived categories of algebras, or a recollement of derived categories of Noetherian algebras, the middle algebra is smooth if and only if so are the algebras on both sides. As a corollary, a triangular matrix algebra is smooth if and only if so are the algebras on diagonal. In section 5, we shall realize triangulated functors in recollements of derived categories of algebras and tensor product algebras as derived functors of the same forms. As applications, from a recollement of derived categories of algebras, we shall obtain in section 6 a triangle on Hochschild complexes due to Keller [22] which can induce a long exact sequence on Hochschild homologies of algebras, and in section 7 three triangles on Hochschild cocomplexes which can induce three long exact sequences on Hochschild cohomologies of algebras. Note that these long exact sequences on Hochschild cohomologies have been widely studied for one-point extensions [16, 14], triangular matrix algebras [8, 30, 15, 9, 6], stratifying ideals [25], homological epimorphisms [36, 39], etc.

2 (Perfect) Recollements

2.1 Recollements

Recall the definition of a recollement of triangulated categories:

Definition 1. (Beilinson-Bernstein-Deligne [5]) Let $\mathcal{T}_1$, $\mathcal{T}$ and $\mathcal{T}_2$ be trian-

gulated categories. A *recollement* of $\mathcal{T}$ relative to $\mathcal{T}_1$ and $\mathcal{T}_2$ is given by

$$\begin{tikzcd} \mathcal{T}_1 \arrow[r, "i_* = i_!"] & \mathcal{T} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^! = j^*"] & \mathcal{T}_2 \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

and denoted by 9-tuple $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ such that

(R1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs of triangulated functors;

(R2) i_*, $j_!$ and j_* are full embeddings;

(R3) $j^! i_* = 0$ (and thus also $i^! j_* = 0$ and $i^* j_! = 0$);

(R4) for each $X \in \mathcal{T}$, there are triangles

$$\begin{array}{l} j_! j^! X \to X \to i_* i^* X \to \\ i_! i^! X \to X \to j_* j^* X \to . \end{array}$$

From now on we focus on recollements of derived categories of algebras. Let A be an algebra. Denote by Proj$A$ (resp. proj$A$) the category of projective (resp. finitely generated projective) A-modules. Denote by $D(A)$ the unbounded derived category of complexes of A-modules. Denote by $K^b(\mathrm{Proj}A)$ (resp. $K^-(\mathrm{Proj}A)$) the homotopy category of bounded (resp. right bounded) complexes of projective A-modules. Let X be an object in $D(A)$. Denote by $X^\perp$ the full subcategory of $D(A)$ consisting of all objects $Y \in D(A)$ such that $\mathrm{Hom}_{D(A)}(X, Y[n]) = 0, \forall\ n \in \mathbb{Z}$. Denote by Tria$X$ the smallest full triangulated subcategory of $D(A)$ which contains X and is closed under small coproducts. We say X is *exceptional* if $\mathrm{Hom}_{D(A)}(X, X[n]) = 0$ for all $n \in \mathbb{Z} \backslash \{0\}$. We say X is *compact* if the functor $\mathrm{Hom}_{D(A)}(X, -)$ preserves small coproduct, or equivalently, X is *perfect*, i.e., isomorphic in $D(A)$ to an object in $K^b(\mathrm{proj}A)$, the homotopy category of bounded complexes of finitely generated projective A-modules. We say X is *self-compact* if $\mathrm{Hom}_{D(A)}(X, -)$ preserves small coproducts in Tria$X$. (ref. [20])

A very important criterion for the right bounded derived category of an algebra to admit a recollement is provided in [23] (cf. [34, Theorem 3]). It was extended and modified to suit for the unbounded derived categories of algebras (ref. [33, Corollary 3.4]), differential graded algebras (ref. [20, Theorem 3.3]) and differential graded categories (ref. [33, Corollary 3.4]).

Theorem 1. (König [23]; Jørgensen [20]; Nicolás-Saorin [33]) *Let A_1, A and A_2 be algebras. Then $D(A)$ admits a recollement relative to $D(A_1)$ and $D(A_2)$ if and only if there are objects X_1 and X_2 in $D(A)$ such that*

(1) $\mathrm{End}_{D(A)}(X_i) \cong A_i$ *as algebras for* $i = 1, 2$*;*
(2) X_2 *(resp.* X_1*) is exceptional and compact (resp. self-compact);*
(3) $X_1 \in X_2^{\perp}$*;*
(4) $X_1^{\perp} \cap X_2^{\perp} = \{0\}$.

An important example of recollements of derived categories of algebras is given by stratifying ideals:

Example 1. (Cline-Parshall-Scott [11]) *Stratifying ideals.* Let A be an algebra, e an idempotent of A, and AeA a stratifying ideal of A, i.e., the multiplication in A induces an isomorphism $Ae \otimes_{eAe} eA \cong AeA$ and $\mathrm{Tor}_n^{eAe}(Ae, eA) = 0$ for all $n \geq 1$. Then there is a recollement $(D(A/AeA), D(A), D(eAe), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ where

$$\begin{array}{ll} i^* = - \otimes_A^L A/AeA, & j_! = - \otimes_{eAe}^L eA, \\ i_* = i_! = - \otimes_{A/AeA}^L A/AeA, & j^! = j^* = - \otimes_A^L Ae, \\ i^! = \mathrm{RHom}_A(A/AeA, -), & j_* = \mathrm{RHom}_{eAe}(Ae, -). \end{array}$$

2.2 Perfect recollements

Sometimes we work on a nicer class of recollements, i.e., the so-called perfect recollements, which correspond to "derived triangular matrix (differential graded) algebras". More precisely, in a perfect recollement of derived categories of algebras, the middle algebra is derived equivalent to a triangular matrix (differential graded) algebra and the algebras on both sides are derived equivalent to the (differential graded) algebras on the diagonal [22].

Definition 2. Let A_1, A and A_2 be algebras. A recollement $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ is said to be *perfect* if $i_* A_1$ is perfect.

From Theorem 1 we can obtain directly the following criterion for the derived category of an algebra to admit a perfect recollement of derived categories of algebras.

Theorem 2. *Let* A_1, A *and* A_2 *be algebras. Then* $D(A)$ *admits a perfect recollement relative to* $D(A_1)$ *and* $D(A_2)$ *if and only if there are objects* $X_i, i = 1, 2,$ *in* $D(A)$ *such that*
(1) $\mathrm{End}_{D(A)}(X_i) \cong A_i$ *as algebras,* $\forall i = 1, 2$*;*
(2) X_i *is exceptional and perfect,* $\forall i = 1, 2$*;*
(3) $X_1 \in X_2^{\perp}$*;*
(4) $X_1^{\perp} \cap X_2^{\perp} = \{0\}$.

Now we provide some examples of either perfect or imperfect recollements of derived categories of algebras:

Example 2. (1) *Derived equivalences.* If the algebras A and B are derived equivalent then $D(A)$ admits a perfect recollement relative to 0 and $D(B)$, or to $D(B)$ and 0. (ref. [24, Theorem 8.3.2])

(2) *Triangular matrix algebras.* Let A_1 and A_2 be algebras, M an A_2-A_1-bimodule, and $A = \left[\begin{smallmatrix} A_1 & 0 \\ M & A_2 \end{smallmatrix}\right]$. Then $X_1 := \left[\begin{smallmatrix} 1_{A_1} & 0 \\ 0 & 0 \end{smallmatrix}\right] A$ and $X_2 := \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1_{A_2} \end{smallmatrix}\right] A$ satisfy all conditions in Theorem 2. Thus there is a perfect recollement of $D(A)$ relative to $D(A_1)$ and $D(A_2)$.

(3) *Perfect stratifying ideals.* Let A be an algebra, e an idempotent of A, and AeA a perfect stratifying ideal of A, i.e., a stratifying ideal which is perfect in $D(A)$. Then there is a perfect recollement of $D(A)$ relative to $D(A/AeA)$ and $D(eAe)$. Note that a triangular matrix algebra always has a projective stratifying ideal.

(4) *Imperfect recollement.* Let A be the infinite Kronecker algebra $\left[\begin{smallmatrix} k & 0 \\ V & k \end{smallmatrix}\right]$, where V is an infinite-dimensional k-vector space. Choose X_2 to be the simple projective A-module and X_1 the other simple A-module. Then X_1 and X_2 satisfy all conditions in Theorem 1. Thus $D(A)$ admits a recollement relative to $D(k)$ and $D(k)$, which is not perfect since X_1 is isomorphic to an object in $K^b(\mathrm{Proj}A)$ but not in $K^b(\mathrm{proj}A)$. (ref. [23, Example 9])

3 Constructions of (perfect) recollements

In this section, we shall provide two constructions of (perfect) recollements : one is via tensor product algebras, the other is via opposite algebras.

3.1 Tensor product algebras

From a recollement of derived categories of algebras, we can obtain recollements of derived categories of tensor product algebras.

Theorem 3. *Let A_1, A and A_2 be algebras, and $D(A)$ admit a (perfect) recollement relative to $D(A_1)$ and $D(A_2)$. Then, for each algebra B, $D(B \otimes A)$ admits a (perfect) recollement relative to $D(B \otimes A_1)$ and $D(B \otimes A_2)$.*

Proof. By Theorem 1, there are objects $X_i, i = 1, 2$, in $D(A)$ such that they satisfy all conditions in Theorem 1. Let $Z_i := B \otimes X_i$ for $i = 1, 2$. Now we show that Z_1 and Z_2 satisfy all conditions in Theorem 1 for tensor product algebras.

Since X_2 is perfect in $D(A)$, Z_2 is perfect in $D(B\otimes A)$. Since X_2 is perfect and exceptional and $\mathrm{End}_{D(A)}(X_2) \cong A_2$ as algebras, we have

$$\begin{array}{rcl}\mathrm{Hom}_{D(B\otimes A)}(Z_2, Z_2[n]) & \cong & H^n(\mathrm{RHom}_{B\otimes A}(B\otimes X_2, B\otimes X_2)) \\ & \cong & H^n(\mathrm{RHom}_B(B, \mathrm{RHom}_A(X_2, B\otimes X_2))) \\ & \cong & H^n(\mathrm{RHom}_A(X_2, B\otimes X_2)) \\ & \cong & \mathrm{Hom}_{D(A)}(X_2, B\otimes X_2[n]) \\ & \cong & B\otimes \mathrm{Hom}_{D(A)}(X_2, X_2[n]) \\ & \cong & \left\{\begin{array}{ll} B\otimes A_2, & \text{if } n=0; \\ 0, & \text{otherwise.}\end{array}\right.\end{array}$$

Thus Z_2 is exceptional and $\mathrm{End}_{D(B\otimes A)}(Z_2) \cong B\otimes A_2$ as algebras.

Since X_1 is self-compact, for any index set Λ, we have

$$\begin{array}{rcl}\mathrm{Hom}_{D(B\otimes A)}(Z_1, Z_1^{(\Lambda)}[n]) & \cong & H^n(\mathrm{RHom}_{B\otimes A}(B\otimes X_1, (B\otimes X_1)^{(\Lambda)})) \\ & \cong & H^n(\mathrm{RHom}_A(X_1, (B\otimes X_1)^{(\Lambda)})) \\ & \cong & \mathrm{Hom}_{D(A)}(X_1, (B\otimes X_1[n])^{(\Lambda)}) \\ & \cong & \mathrm{Hom}_{D(A)}(X_1, B\otimes X_1[n])^{(\Lambda)} \\ & \cong & H^n(\mathrm{RHom}_A(X_1, B\otimes X_1))^{(\Lambda)} \\ & \cong & H^n(\mathrm{RHom}_{B\otimes A}(B\otimes X_1, B\otimes X_1))^{(\Lambda)} \\ & \cong & \mathrm{Hom}_{D(B\otimes A)}(Z_1, Z_1[n])^{(\Lambda)}.\end{array}$$

Thus Z_1 is self-compact.

Since X_1 is self-compact and exceptional and $\mathrm{End}_{D(A)}(X_1) \cong A_1$ as algebras, we have

$$\begin{array}{rcl}\mathrm{Hom}_{D(B\otimes A)}(Z_1, Z_1[n]) & \cong & H^n(\mathrm{RHom}_{B\otimes A}(B\otimes X_1, B\otimes X_1)) \\ & \cong & H^n(\mathrm{RHom}_A(X_1, B\otimes X_1)) \\ & \cong & \mathrm{Hom}_{D(A)}(X_1, B\otimes X_1[n]) \\ & \cong & B\otimes \mathrm{Hom}_{D(A)}(X_1, X_1[n]) \\ & \cong & \left\{\begin{array}{ll} B\otimes A_1, & \text{if } n=0; \\ 0, & \text{otherwise.}\end{array}\right.\end{array}$$

Thus Z_1 is exceptional and $\mathrm{End}_{D(B\otimes A)}(Z_1) \cong B\otimes A_1$ as algebras.

Since X_2 is perfect and $X_1 \in X_2^{\perp}$, we have

$$\begin{array}{rcl}\mathrm{Hom}_{D(B\otimes A)}(Z_2, Z_1[n]) & \cong & H^n(\mathrm{RHom}_{B\otimes A}(B\otimes X_2, B\otimes X_1)) \\ & \cong & H^n(\mathrm{RHom}_A(X_2, B\otimes X_1)) \\ & \cong & \mathrm{Hom}_{D(A)}(X_2, B\otimes X_1[n]) \\ & \cong & B\otimes \mathrm{Hom}_{D(A)}(X_2, X_1[n]) = 0\end{array}$$

for all $n\in\mathbb{Z}$. Thus $Z_1 \in Z_2^{\perp}$.

For any $Z \in Z_1^{\perp} \cap Z_2^{\perp}$, we have

$$\begin{array}{rcl} 0 & = & \mathrm{Hom}_{D(B\otimes A)}(Z_i, Z[n]) \\ & \cong & H^n(\mathrm{RHom}_{B\otimes A}(B\otimes X_i, Z)) \\ & \cong & H^n(\mathrm{RHom}_A(X_i, Z)) \\ & \cong & \mathrm{Hom}_{D(A)}(X_i, Z[n]) \end{array}$$

for all $n \in \mathbb{Z}$ and $i = 1, 2$. Thus $Z \in X_1^{\perp} \cap X_2^{\perp} = \{0\}$. Hence $Z_1^{\perp} \cap Z_2^{\perp} = \{0\}$.

Now we have shown that Z_1 and Z_2 satisfy all conditions in Theorem 1 for tensor product algebras. By Theorem 1, we are done. The statement on perfect recollement can be obtained simultaneously. □

3.2 Opposite algebras

From a perfect recollement of derived categories of algebras, we can obtain a perfect recollement of derived categories of opposite algebras.

Theorem 4. *Let A_1, A and A_2 be algebras, and $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ a perfect recollement. Then $D(A^{\mathrm{op}})$ admits a perfect recollement relative to $D(A_2^{\mathrm{op}})$ and $D(A_1^{\mathrm{op}})$.*

Proof. The objects $X_1 := i_*A_1$ and $X_2 := j_!A_2$ in $D(A)$ satisfy all conditions in Theorem 2. Let $Z_i = \mathrm{RHom}_A(X_i, A), i = 1, 2$. Now we show that Z_1 and Z_2 satisfy all conditions in Theorem 2 for opposite algebras.

Since X_i is perfect in $D(A)$, Z_i is perfect in $D(A^{\mathrm{op}})$. Since X_i is perfect and exceptional and $\mathrm{End}_{D(A)}(X_i) \cong A_i$ as algebras, we have

$$\begin{array}{rcl} \mathrm{Hom}_{D(A^{\mathrm{op}})}(Z_i, Z_i[n]) & \cong & H^n(\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(X_i, A), \mathrm{RHom}_A(X_i, A))) \\ & \cong & H^n(\mathrm{RHom}_A(X_i, X_i)) \\ & \cong & \mathrm{Hom}_{D(A)}(X_i, X_i[n]) \\ & \cong & \begin{cases} A_i, & \text{if } n = 0; \\ 0, & \text{otherwise.} \end{cases} \end{array}$$

Thus Z_i is exceptional and $\mathrm{End}_{D(A^{\mathrm{op}})}(Z_i) \cong A_i^{\mathrm{op}}$ as algebras for $i = 1, 2$.

Since X_1 and X_2 are perfect and $X_1 \in X_2^{\perp}$, we have

$$\begin{array}{rcl} \mathrm{Hom}_{D(A^{\mathrm{op}})}(Z_1, Z_2[n]) & \cong & H^n(\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(X_1, A), \mathrm{RHom}_A(X_2, A))) \\ & \cong & H^n(\mathrm{RHom}_A(X_2, X_1)) \\ & \cong & \mathrm{Hom}_{D(A)}(X_2, X_1[n]) = 0 \end{array}$$

for all $n \in \mathbb{Z}$. Thus $Z_2 \in Z_1^{\perp}$.

For all $Z \in Z_1^\perp \cap Z_2^\perp$ and $n \in \mathbb{Z}$, since $X_i, i = 1, 2$, are perfect, we have

$$\begin{array}{rcl} 0 & = & \mathrm{Hom}_{D(A^{\mathrm{op}})}(Z_i, Z[n]) \\ & \cong & H^n(\mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(X_i, A), Z)) \\ & \cong & H^n(X_i \otimes_A^L Z) \end{array}$$

i.e., $X_i \otimes_A^L Z = 0$ in $D(k)$. Thus, for all $W \in D(k)$ and $n \in \mathbb{Z}$, we have

$$\begin{array}{rcl} H^n(i^!\mathrm{RHom}_k(Z, W)) & \cong & H^n(\mathrm{RHom}_{A_1}(A_1, i^!\mathrm{RHom}_k(Z, W))) \\ & \cong & \mathrm{Hom}_{D(A_1)}(A_1, i^!\mathrm{RHom}_k(Z, W)[n]) \\ & \cong & \mathrm{Hom}_{D(A)}(i_*A_1, \mathrm{RHom}_k(Z, W)[n]) \\ & = & \mathrm{Hom}_{D(A)}(X_1, \mathrm{RHom}_k(Z, W)[n]) \\ & \cong & \mathrm{Hom}_{D(k)}(X_1 \otimes_A^L Z, W[n]) = 0, \end{array}$$

i.e., $i^!\mathrm{RHom}_k(Z, W) = 0$ in $D(A_1)$. Similarly, we have

$$\begin{array}{rcl} H^n(j^*\mathrm{RHom}_k(Z, W)) & \cong & H^n(\mathrm{RHom}_{A_2}(A_2, j^*\mathrm{RHom}_k(Z, W))) \\ & \cong & \mathrm{Hom}_{D(A_2)}(A_2, j^*\mathrm{RHom}_k(Z, W)[n]) \\ & \cong & \mathrm{Hom}_{D(A)}(j_!A_2, \mathrm{RHom}_k(Z, W)[n]) \\ & = & \mathrm{Hom}_{D(A)}(X_2, \mathrm{RHom}_k(Z, W)[n]) \\ & \cong & \mathrm{Hom}_{D(k)}(X_2 \otimes_A^L Z, W[n]) = 0, \end{array}$$

i.e., $j^*\mathrm{RHom}_k(Z, W) = 0$ in $D(A_2)$. In the triangle

$$i_!i^!\mathrm{RHom}_k(Z, W) \to \mathrm{RHom}_k(Z, W) \to j_*j^*\mathrm{RHom}_k(Z, W) \to,$$

both sides are zero. Thus $\mathrm{RHom}_k(Z, W) = 0$ for all $W \in D(k)$. Hence $Z = 0$ in $D(A^{\mathrm{op}})$. Therefore, $Z_1^\perp \cap Z_2^\perp = \{0\}$.

Now we have shown that Z_1 and Z_2 satisfy all conditions in Theorem 2 for opposite algebras. By Theorem 2, we are done. □

4 Recollements and smoothness

In this section, we shall apply two constructions provided in section 3 to study the relations between recollements of derived categories of algebras and smoothness of algebras, i.e., finiteness of Hochschild dimensions of algebras. For this, we need to know the relation between recollements of derived categories of algebras and global dimensions of algebras.

4.1 Recollements and global dimensions

The following result is due to König:

Theorem 5. (König [23, Corollary 5]) *Let A_1, A and A_2 be algebras, and $D(A)$ admit a perfect recollement relative to $D(A_1)$ and $D(A_2)$. Then A is of finite global dimension if and only if so are A_1 and A_2.*

Proof. Denote by $D^-(A)$ the derived category of complexes of A-modules with right bounded cohomologies. By Theorem 2 and [23, Theorem 1] or [34, Theorem 3], we have a restricted recollement of $D^-(A)$ relative to $D^-(A_1)$ and $D^-(A_2)$. By [23, Corollary 5], we are done. □

Recently, the following more general result is obtained:

Theorem 6. (Angeleri Hügel-König-Liu-Yang [2]) *Let A_1, A and A_2 be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then A is of finite global dimension if and only if so are A_1 and A_2.*

4.2 Recollements and smoothness

Let A be an algebra and $A^e := A^{\mathrm{op}} \otimes A$ its enveloping algebra. The *Hochschild dimension* of A is the projective dimension of A as a left or right A^e-module. The Hochschild dimensions of algebras were studied very earlier [7]. An algebra A is of Hochschild dimension 0 if and only if A^e is semisimple [7, Theorem 7.9]. In case A is finitely generated, A is of Hochschild dimension 0 if and only if A is separable [7, Theorem 7.10]. The algebras of Hochschild dimension ≤ 1 are called *quasi-free* or *formally smooth* [12, 26]. An algebra A is said to be *smooth* if it has finite Hochschild dimension, i.e., $\mathrm{pd}_{A^e} A < \infty$ (ref. [40]), or equivalently, A is isomorphic to an object in $K^b(\mathrm{Proj} A^e)$. It follows from [7, Chap. IX, Proposition 7.5, 7.6] and [13, Proposition 2] that A is smooth if and only if $\mathrm{gl.dim} A^e < \infty$.

Remark 1.

(1) *Sometimes* $\mathrm{gl.dim} A < \infty \Leftrightarrow \mathrm{gl.dim} A^e < \infty$: Let A be either a commutative Noetherian algebra over a perfect field k, or a finite-dimensional k-algebra such that the factor algebra A/J of A modulo its Jacobson radical J is separable. Then $\mathrm{gl.dim} A < \infty$ if and only if $\mathrm{gl.dim} A^e < \infty$. (ref. [18, Theorem 2.1] and [3, Theorem 16])

(2) *In general* $\mathrm{gl.dim} A < \infty \nRightarrow \mathrm{gl.dim} A^e < \infty$: Let A be a finite inseparable field extension of an imperfect field k. Then $\mathrm{gl.dim} A = 0$. However, $\mathrm{gl.dim} A^e = \infty$, since $A \otimes_k A$ is not semisimple. (ref. [4, Page 65, Remark])

Let the algebras A and B be derived equivalent. Then by [37, Proposition 9.1] and [38, Theorem 2.1] we know A^e and B^e are derived equivalent. Thus gl.dim$A^e < \infty$ if and only if gl.dim$B^e < \infty$ (ref. [24, p.37]). Hence, A is smooth if and only if so is B, i.e., the smoothness of algebras is invariant under derived equivalences. More general, we have the following result:

Theorem 7. *Let A_1, A and A_2 be algebras, and $D(A)$ admit a perfect recollement relative to $D(A_1)$ and $D(A_2)$. Then A is smooth if and only if so are A_1 and A_2.*

Proof. By Theorem 3, we have a perfect recollement of $D(A^{\mathrm{op}} \otimes A)$ relative to $D(A^{\mathrm{op}} \otimes A_1)$ and $D(A^{\mathrm{op}} \otimes A_2)$. It follows from Theorem 5 that gl.dim$A^e < \infty$ if and only if gl.dim$A^{\mathrm{op}} \otimes A_i < \infty$ for all $i = 1, 2$. By Theorem 4 and Theorem 3, we have a perfect recollement of $D(A^{\mathrm{op}} \otimes A_i)$ relative to $D(A_2^{\mathrm{op}} \otimes A_i)$ and $D(A_1^{\mathrm{op}} \otimes A_i)$. It follows from Theorem 5 that gl.dim$A^{\mathrm{op}} \otimes A_i < \infty$ if and only if gl.dim$A_j^{\mathrm{op}} \otimes A_i < \infty$ for all $i, j = 1, 2$. Therefore, gl.dim$A^e < \infty$ if and only if gl.dim$A_i^e < \infty$ for all $i = 1, 2$, by [13, Proposition 2]. $\square$

Theorem 7 can be applied to judge the smoothness of some algebras or construct some smooth algebras. For instance, when applied to triangular matrix algebras, we have the following result:

Corollary 1. *Let A_1 and A_2 be algebras, M an A_2-A_1-bimodule, and $A = \left[\begin{smallmatrix} A_1 & 0 \\ M & A_2 \end{smallmatrix}\right]$. Then A is smooth if and only if so are A_1 and A_2.*

Proof. It follows from Example 2 (2) and Theorem 7. $\square$

Remark 2. An algebra A is said to be *homotopically smooth* if A is compact in $D(A^e)$, i.e., A is isomorphic in $D(k)$ to an object in $K^b(\mathrm{proj}A^e)$ (ref. [27]). By Corollary 1, we know the infinite Kronecker algebra (ref. Example 2 (4)) is smooth. However, it is not homotopically smooth, because the finitely generated projective A^e-module resolution of A would induce a finitely generated projective A-module resolution of the nonprojective simple A-module. Hence, Corollary 1, thus Theorem 7, is not correct for homotopically smoothness.

We don't know whether Theorem 7 holds for any recollement of derived categories of algebras? However, it does hold when restricted to Noetherian algebras.

Theorem 8. *Let A_1, A and A_2 be Noetherian algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then A is smooth if and only if so are A_1 and A_2.*

Proof. Necessity. Since A is smooth, by [7, Chap. IX, Proposition 7.6], we have gl.dim$A < \infty$. It follows from Theorem 6 that gl.dim$A_i < \infty$ for all $i = 1, 2$. Since A_i is Noetherian, by [3, Corollary 5], we have gl.dim$A_i^{\mathrm{op}} =$ gl.dim$A_i < \infty$ for all $i = 1, 2$. It follows from [13, Proposition 2] that gl.dim$A_i^{\mathrm{op}} \otimes A < \infty$ for all $i = 1, 2$. By Theorem 3, we have a recollement of $D(A_i^{\mathrm{op}} \otimes A)$ relative to $D(A_i^{\mathrm{op}} \otimes A_1)$ and $D(A_i^{\mathrm{op}} \otimes A_2)$. It follows from Theorem 6 that gl.dim$A_i^{\mathrm{op}} \otimes A_j < \infty$ for all $i, j = 1, 2$. Therefore, gl.dim$A_i^e < \infty$, i.e., A_i is smooth, for all $i = 1, 2$.

Sufficiency. Since A_i is smooth, by [7, Chap. IX, Proposition 7.6], we have gl.dim$A_i < \infty$ for all $i = 1, 2$. It follows from Theorem 6 that gl.dim$A < \infty$. Since A is Noetherian, by [3, Corollary 5], we have gl.dim$A^{\mathrm{op}} =$ gl.dim$A < \infty$. It follows from [13, Proposition 2] that gl.dim$A^{\mathrm{op}} \otimes A_i < \infty$ for all $i = 1, 2$. By Theorem 3, we have a recollement of $D(A^{\mathrm{op}} \otimes A)$ relative to $D(A^{\mathrm{op}} \otimes A_1)$ and $D(A^{\mathrm{op}} \otimes A_2)$. It follows from Theorem 6 that gl.dim$A^{\mathrm{op}} \otimes A < \infty$, i.e., A is smooth. $\square$

5 Recollements and derived functors

In this section, we shall realize triangulated functors in recollements of derived categories of algebras and tensor product algebras as derived functors of the same forms in the spirit of [38].

Lemma 1. *Let A_1, A and A_2 be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then*

(1) *there is a recollement $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ such that*

$$\begin{array}{ll} & j_! = - \otimes^L_{A_2} Y_2, \\ i_* = i_! = - \otimes^L_{A_1} Y_1, & j^! = j^* = \mathrm{RHom}_A(Y_2, -), \\ i^! = \mathrm{RHom}_A(Y_1, -), & j_* = \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -), \end{array}$$

and

(2) *for each algebra B, there is a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ such that*

$$\begin{array}{ll} & J_! = - \otimes^L_{A_2} Y_2, \\ I_* = I_! = - \otimes^L_{A_1} Y_1, & J^! = J^* = \mathrm{RHom}_A(Y_2, -), \\ I^! = \mathrm{RHom}_A(Y_1, -), & J_* = \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -), \end{array}$$

for some $Y_i \in D(A_i^{\mathrm{op}} \otimes A)$, $i = 1, 2$.

Proof. It follows from Theorem 1 that there are objects $X_i, i = 1, 2$, in $D(A)$ such that they satisfy all conditions in Theorem 1. Clearly, we may assume that X_1 and X_2 are homotopically projective. Since X_i is exceptional, it follows from [24, 8.3.1] that there exists $Y_i \in D(A_i^{\mathrm{op}} \otimes A)$ such that the derived tensor functor $- \otimes^L_{A_i} Y_i : D(A_i) \to D(A)$ sends A_i to X_i for all $i = 1, 2$.

(1) By [32, Theorem 2.8], we have a recollement $(X_2^{\perp}, D(A), D(A_2), i'^*, i'_* = i'_!, i'^!, j_!, j^! = j^*, j_*)$ where $j_! = - \otimes^L_{A_2} Y_2, j^! = j^* = \mathrm{RHom}_A(Y_2, -), j_* = \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -)$, and $i'_* = i'_!$ is the natural embedding.

Since X_1 is self-compact, it is a compact generator in $\mathrm{Tria}X_1$. It follows from [21, Lemma 4.2] that the functor $- \otimes^L_{A_1} Y_1 : D(A_1) \to D(A)$ is fully faithful and its essential image is $\mathrm{Tria}X_1$. From $X_1 \in X_2^{\perp}$ and $X_1^{\perp} \cap X_2^{\perp} = \{0\}$ we can obtain $\mathrm{Tria}X_1 = X_2^{\perp}$ (ref. [20, Proof of Theorem 3.3]). Thus we have a recollement $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ with described five triangulated functors.

(2) Repeat the proof above, mutatis mutandis, we can obtain a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ such that

$$\begin{array}{ll} & J_! = - \otimes^L_{B \otimes A_2} (B \otimes Y_2), \\ I_* = I_! = - \otimes^L_{B \otimes A_1} (B \otimes Y_1), & J^! = J^* = \mathrm{RHom}_{B \otimes A}(B \otimes Y_2, -), \\ I^! = \mathrm{RHom}_{B \otimes A}(B \otimes Y_1, -), & J_* = \mathrm{RHom}_{B \otimes A_2}(\mathrm{RHom}_{B \otimes A}(B \otimes Y_2, B \otimes A), -). \end{array}$$

Note that all conditions on $B \otimes X_i$ needed here have been checked already in the proof of Theorem 3.

Clearly, $I_* = I_! = - \otimes^L_{B \otimes A_1} (B \otimes Y_1) \cong (- \otimes^L_B B) \otimes^L_{A_1} Y_1 \cong - \otimes^L_{A_1} Y_1$. Similarly, $J_! \cong - \otimes^L_{A_2} Y_2$. We also have $I^! = \mathrm{RHom}_{B \otimes A}(B \otimes Y_1, -) \cong \mathrm{RHom}_B(B, \mathrm{RHom}_A(Y_1, -)) \cong \mathrm{RHom}_A(Y_1, -)$. Similarly, $J^! = J^* \cong \mathrm{RHom}_A(Y_2, -)$. Moreover, since Y_2 is perfect in $D(A)$, we have

$$\begin{aligned} J_* &= \mathrm{RHom}_{B \otimes A_2}(\mathrm{RHom}_{B \otimes A}(B \otimes Y_2, B \otimes A), -) \\ &\cong \mathrm{RHom}_{B \otimes A_2}(\mathrm{RHom}_A(Y_2, B \otimes A), -) \\ &\cong \mathrm{RHom}_{B \otimes A_2}(B \otimes \mathrm{RHom}_A(Y_2, A), -) \\ &\cong \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -). \end{aligned}$$

Thus we can obtain a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ with described triangulated functors. $\square$

Remark 3. Two recollements of triangulated categories $(\mathcal{T}_1, \mathcal{T}, \mathcal{T}_2, i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ and $(\mathcal{T}_1', \mathcal{T}, \mathcal{T}_2', i'^*, i'_* = i'_!, i'^!, j'_!, j'^! = j'^*, j'_*)$ are said to be *equivalent* if $(\mathrm{Im}j_!, \mathrm{Im}i_*, \mathrm{Im}j_*) = (\mathrm{Im}j'_!, \mathrm{Im}i'_*, \mathrm{Im}j'_*)$, where Im denotes the

essential image of the functor [33]. It is easy to see that the resulting recollement in the first part of Lemma 1 is equivalent to the known one. Indeed, $\mathrm{Im} j_! = \mathrm{Tria} X_2 = \mathrm{Tria} Y_2 = \mathrm{Im}(- \otimes^L_{A_2} Y_2)$, $\mathrm{Im} i_* = \mathrm{Tria} X_1 = \mathrm{Tria} Y_1 = \mathrm{Im}(- \otimes^L_{A_1} Y_1)$ and $\mathrm{Im} j_* = X_1^\perp = Y_1^\perp = \mathrm{Im}(\mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -))$.

Lemma 2. *Let $\mathcal{T}$ be a triangulated category with translation functor* $[1]$, *$F_i, G_i : \mathcal{T} \to \mathcal{T}$, $i = 1, 2, 3$, six triangulated functors, $\alpha_j : F_j \to F_{j+1}$, $\beta_j : G_j \to G_{j+1}$, $\gamma_j : F_j \to G_j$, $j = 1, 2$, six natural transformations, such that $\gamma_2\alpha_1 = \beta_1\gamma_1$, $F_1X \overset{\alpha_{1X}}{\to} F_2X \overset{\alpha_{2X}}{\to} F_3X \to$ and $G_1X \overset{\beta_{1X}}{\to} G_2X \overset{\beta_{2X}}{\to} G_3X \to$ are triangles for all $X \in \mathcal{T}$, and* $\mathrm{Hom}_{\mathcal{T}}(F_1, G_3[-1]) = 0$. *Then there is a natural transformation $\gamma_3 : F_3 \to G_3$ such that $\gamma_3\alpha_2 = \beta_2\gamma_2$.*

Proof. For each $X \in \mathcal{T}$, by assumption, we have a commutative diagram

$$\begin{tikzcd}
F_1X \arrow[r, "\alpha_{1X}"] \arrow[d, "\gamma_{1X}"] & F_2X \arrow[r, "\alpha_{2X}"] \arrow[d, "\gamma_{2X}"] & F_3X \arrow[r] & {} \\
G_1X \arrow[r, "\beta_{1X}"] & G_2X \arrow[r, "\beta_{2X}"] & G_3X \arrow[r] & {}
\end{tikzcd}$$

By an axiom of triangulated category, there exists a morphism $\gamma_{3X} : F_3X \to G_3X$ such that $\gamma_{3X}\alpha_{2X} = \beta_{2X}\gamma_{2X}$.

For each morphism $f : X \to Y$ in $\mathcal{T}$, we have commutative diagrams

$$\begin{tikzcd}
F_1X \arrow[r, "\alpha_{1X}"] \arrow[d, "G_1(f)\gamma_{1X}"] & F_2X \arrow[r, "\alpha_{2X}"] \arrow[d, "G_2(f)\gamma_{2X}"] & F_3X \arrow[r] \arrow[d, "G_3(f)\gamma_{3X}"] & {} \\
G_1Y \arrow[r, "\beta_{1Y}"] & G_2Y \arrow[r, "\beta_{2Y}"] & G_3Y \arrow[r] & {}
\end{tikzcd}$$

and

$$\begin{tikzcd}
F_1X \arrow[r, "\alpha_{1X}"] \arrow[d, "\gamma_{1Y}F_1(f)"] & F_2X \arrow[r, "\alpha_{2X}"] \arrow[d, "\gamma_{2Y}F_2(f)"] & F_3X \arrow[r] \arrow[d, "\gamma_{3Y}F_3(f)"] & {} \\
G_1Y \arrow[r, "\beta_{1Y}"] & G_2Y \arrow[r, "\beta_{2Y}"] & G_3Y \arrow[r] & {}
\end{tikzcd}.$$

Since $G_1(f)\gamma_{1X} = \gamma_{1Y}F_1(f)$ and $G_2(f)\gamma_{2X} = \gamma_{2Y}F_2(f)$, by [5, Proposition 1.1.9] we have $G_3(f)\gamma_{3X} = \gamma_{3Y}F_3(f)$.

Thus γ_3 is a natural transformation such that $\gamma_3\alpha_2 = \beta_2\gamma_2$. □

Now we can realize all six triangulated functors in a recollement of derived categories of algebras or tensor product algebras as derived functors:

Theorem 9. *Let A_1, A and A_2 be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then*

(1) *there is a recollement* $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ *such that*

$$\begin{array}{ll} i^* = - \otimes^L_A Y, & j_! = - \otimes^L_{A_2} Y_2, \\ i_* = i_! = - \otimes^L_{A_1} Y_1, & j^! = j^* = \mathrm{RHom}_A(Y_2, -), \\ i^! = \mathrm{RHom}_A(Y_1, -), & j_* = \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -), \end{array}$$

and

(2) *for each algebra B, there is a recollement* $(D(B\otimes A_1), D(B\otimes A), D(B\otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ *such that*

$$\begin{array}{ll} I^* = - \otimes^L_A Y, & J_! = - \otimes^L_{A_2} Y_2, \\ I_* = I_! = - \otimes^L_{A_1} Y_1, & J^! = J^* = \mathrm{RHom}_A(Y_2, -), \\ I^! = \mathrm{RHom}_A(Y_1, -), & J_* = \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -), \end{array}$$

for some $Y \in D(A^{\mathrm{op}} \otimes A_1)$ and $Y_i \in D(A_i^{\mathrm{op}} \otimes A)$, $i = 1, 2$.

Proof. (1) By Lemma 1, we may assume that all triangulated functors but i^* in the recollement are as required. For each $X \in D(A)$, we have a triangle in $D(A)$:

$$j_!j^!X \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} i_*i^*X \to$$

where ε is the counit of the adjoint pair $(j_!, j^!)$ and η is the unit of the adjoint pair (i^*, i_*).

By Lemma 1, we have a recollement $(D(A^{\mathrm{op}} \otimes A_1), D(A^{\mathrm{op}} \otimes A), D(A^{\mathrm{op}} \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ such that

$$\begin{array}{ll} & J_! = - \otimes^L_{A_2} Y_2, \\ I_* = I_! = - \otimes^L_{A_1} Y_1, & J^! = J^* = \mathrm{RHom}_A(Y_2, -), \\ I^! = \mathrm{RHom}_A(Y_1, -), & J_* = \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -). \end{array}$$

for some $Y_i \in D(A_i^{\mathrm{op}} \otimes A)$, $i = 1, 2$.

Let $Y := I^*A \in D(A^{\mathrm{op}} \otimes A_1)$. Then we have a triangle in $D(A^e)$:

$$J_!J^!A = \mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2 \xrightarrow{E_A} A \xrightarrow{H_A} I_*I^*A = Y \otimes^L_{A_1} Y_1 \to$$

where E is the counit of the adjoint pair $(J_!, J^!)$ and H is the unit of the adjoint pair (I^*, I_*).

Since the the functor $X \otimes_A^L -$ is triangulated, we have a triangle in $D(A)$:

$$X \otimes_A^L \mathrm{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2 \overset{X\otimes_A^L E_A}{\to} X \otimes_A^L A \overset{X\otimes_A^L H_A}{\to} X \otimes_A^L Y \otimes_{A_1}^L Y_1 \to .$$

Clearly, there are natural isomorphisms $\gamma_1 : - \otimes_A^L \mathrm{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2 \to \mathrm{RHom}_A(Y_2, -) \otimes_{A_2}^L Y_2 = j_! j^!$ and $\gamma_2 : - \otimes_A^L A \to -$ such that $\varepsilon\gamma_1 = \gamma_2(- \otimes_A^L E_A)$. Since

$$\begin{aligned} & \mathrm{Hom}_{D(A)}(- \otimes_A^L \mathrm{RHom}_A(Y_2, A) \otimes_{A_2}^L Y_2, i_* i^*[-1](?)) \\ \cong\ & \mathrm{Hom}_{D(A)}(j_! j^!(-), i_* i^*[-1](?)) \\ \cong\ & \mathrm{Hom}_{D(A)}(j^!(-), j^! i_* i^*[-1](?)) = 0, \end{aligned}$$

by Lemma 2 there is a natural isomorphism $\gamma_3 : - \otimes_A^L Y \otimes_{A_1}^L Y_1 \to i_* i^*$ such that $\gamma_3(- \otimes_A^L H_A) = \eta\gamma_2$. Hence $i^* \cong i^! i_* i^* \cong i^!(- \otimes_A^L Y \otimes_{A_1}^L Y_1) \cong i^! i_!(- \otimes_A^L Y) \cong - \otimes_A^L Y$. Therefore, we may replace i^* with $- \otimes_A^L Y$ and obtain a recollement as required. (cf. The proof of [32, Theorem 2.8])

(2) Repeat the proof above, mutatis mutandis, we can obtain a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ as required. $\square$

In Theorem 9, all six triangulated functors in a recollement of derived categories of algebras or tensor product algebras are realized as derived functors, which are given by three bimodule complexes Y, Y_1 and Y_2. In the following, we show that Y and Y_2 are enough.

Definition 3. A recollement $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ is said to be *standard* and *given by* $Y \in D(A^{\mathrm{op}} \otimes A_1)$ and $Y_2 \in D(A_2^{\mathrm{op}} \otimes A)$ if $i^* \cong - \otimes_A^L Y$ and $j_! \cong - \otimes_{A_2}^L Y_2$.

By Theorem 9 and Remark 3, we know any recollement of derived categories of algebras is equivalent to a standard one.

Theorem 10. *Let A_1, A and A_2 be algebras, and $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ a standard recollement given by $Y \in D(A^{\mathrm{op}} \otimes A_1)$ and $Y_2 \in D(A_2^{\mathrm{op}} \otimes A)$. Then*

(1)

$$\begin{aligned} & i^* \cong - \otimes_A^L Y, && j_! \cong - \otimes_{A_2}^L Y_2, \\ & i_* = i_! \cong \mathrm{RHom}_{A_1}(Y, -), && j^! = j^* \cong \mathrm{RHom}_A(Y_2, -), \\ & i^! \cong \mathrm{RHom}_A(\mathrm{RHom}_{A_1}(Y, A_1), -), && j_* \cong \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -), \end{aligned}$$

and

(2) *for each algebra B, there is a recollement* $(D(B\otimes A_1), D(B\otimes A), D(B\otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ *such that*

$$\begin{array}{ll} I^* \cong - \otimes^L_A Y, & J_! \cong - \otimes^L_{A_2} Y_2, \\ I_* = I_! \cong \mathrm{RHom}_{A_1}(Y, -), & J^! = J^* \cong \mathrm{RHom}_A(Y_2, -), \\ I^! \cong \mathrm{RHom}_A(\mathrm{RHom}_{A_1}(Y, A_1), -), & J_* \cong \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -). \end{array}$$

Proof. (1) Since i_* is a left adjoint of $i^!$, it commutes with small coproduct. The functor $- \otimes^L_A Y \cong i^*$ has a right adjoint i_* which commutes with small coproduct, thus Y is perfect in $D(A_1)$. Therefore, $\mathrm{RHom}_{A_1}(Y, -) \cong - \otimes^L_{A_1} \mathrm{RHom}_{A_1}(Y, A_1)$ (ref. [32, Lemma 2.6]). Since the right adjoint is unique up to natural isomorphism, we can take $i_* = i_!$ and $i^!$ as required. Similar for $j^! = j^*$ and j_*.

(2) The proof of Lemma 1 implies that, for each algebra B, there is a recollement $(D(B \otimes A_1), D(B \otimes A), D(B \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ such that

$$\begin{array}{ll} & J_! \cong - \otimes^L_{A_2} Y_2, \\ I_* = I_! \cong - \otimes^L_{A_1} \mathrm{RHom}_{A_1}(Y, A_1), & J^! = J^* \cong \mathrm{RHom}_A(Y_2, -), \\ I^! \cong \mathrm{RHom}_A(\mathrm{RHom}_{A_1}(Y, A_1), -), & J_* \cong \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -). \end{array}$$

Since I^* is a left adjoint of $- \otimes^L_{A_1} \mathrm{RHom}_{A_1}(Y, A_1) \cong \mathrm{RHom}_{A_1}(Y, -)$ and the left adjoint is unique up to natural isomorphism, we have $I^* \cong - \otimes^L_A Y$. $\square$

6 Recollements and Hochschild homology

In this section, we shall apply the results obtained in section 5 to study the relation between recollements of derived categories of algebras and Hochschild homology of algebras, which had been clarified by Keller in [22]. Recall that the *n-th Hochschild homology* of an algebra A is $HH_n(A) := \mathrm{Tor}^{A^e}_n(A, A) \cong H^{-n}(A \otimes^L_{A^e} A)$. Note that in $D(k)$ the complex $A \otimes^L_{A^e} A$ is isomorphic to the Hochschild complex of A. The following result is due to Keller, which is a corollary of [22, Theorem 3.1] (ref. [22, Remarks 3.2 (a)]). Here, we apply Theorem 9 to give a direct proof.

Theorem 11. (Keller [22]) *Let A, A_1 and A_2 be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then there is a triangle in $D(k)$:*

$$A_2 \otimes^L_{A_2^e} A_2 \to A \otimes^L_{A^e} A \to A_1 \otimes^L_{A_1^e} A_1 \to .$$

Proof. Applying Theorem 9 to the case $B = A^{\rm op}$, we obtain a recollement $(D(A^{\rm op} \otimes A_1), D(A^{\rm op} \otimes A), D(A^{\rm op} \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$. Furthermore, we have a triangle in $D(A^e)$:

$$J_! J^! A = \mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2 \to A \to I_* I^* A = Y \otimes^L_{A_1} Y_1 \to .$$

The derived tensor functor $- \otimes^L_{A^e} A : D(A^e) \to D(k)$ sends this triangle to a triangle in $D(k)$:

$$(\mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2) \otimes^L_{A^e} A \to A \otimes^L_{A^e} A \to (Y \otimes^L_{A_1} Y_1) \otimes^L_{A^e} A \to .$$

Its left hand side

$$\begin{aligned} (\mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2) \otimes^L_{A^e} A &\cong (Y_2 \otimes^L_A \mathrm{RHom}_A(Y_2, A)) \otimes^L_{A_2^e} A_2 \\ &\cong J^! J_! A_2 \otimes^L_{A_2^e} A_2 \\ &\cong A_2 \otimes^L_{A_2^e} A_2 \end{aligned}$$

and its right hand side

$$(Y \otimes^L_{A_1} Y_1) \otimes^L_{A^e} A \cong (Y_1 \otimes^L_A Y) \otimes^L_{A_1^e} A_1 \cong I^* I_* A_1 \otimes^L_{A_1^e} A_1 \cong A_1 \otimes^L_{A_1^e} A_1,$$

thus we have a triangle in $D(k)$:

$$A_2 \otimes^L_{A_2^e} A_2 \to A \otimes^L_{A^e} A \to A_1 \otimes^L_{A_1^e} A_1 \to .$$

□

From the triangle in Theorem 11, by taking cohomologies, we can obtain a long exact sequence on the Hochschild homologies of the algebras:

Corollary 2. (Keller [22]) *Let A, A_1 and A_2 be algebras, and $D(A)$ admit a recollement relative to $D(A_1)$ and $D(A_2)$. Then there is a long exact sequence on the Hochschild homologies of these algebras*

$$\cdots \to HH_{n+1}(A_1) \to HH_n(A_2) \to HH_n(A) \to HH_n(A_1) \to \cdots .$$

For perfect recollements of derived categories of algebras, we have the following stronger conclusion:

Theorem 12. (Keller [22, Remarks 3.2 (b)]) *Let A, A_1 and A_2 be algebras, and $D(A)$ admit a perfect recollement relative to $D(A_1)$ and $D(A_2)$. Then*

$$A \otimes^L_{A^e} A \cong (A_1 \otimes^L_{A_1^e} A_1) \oplus (A_2 \otimes^L_{A_2^e} A_2).$$

Indeed, perfect recollements of derived categories of algebras correspond to "derived triangulated matrix (differential graded) algebras". Then Kadison's method works here (ref. [29, 1.2.15]).

7 Recollements and Hochschild cohomology

In this section, we shall apply the results obtained in section 5 to observe the relations between recollements of derived categories of algebras and Hochschild cohomology of algebras. Recall that the *n-th Hochschild cohomology* of an algebra A is $HH^n(A) := \mathrm{Ext}^n_{A^e}(A, A) \cong H^n(\mathrm{RHom}_{A^e}(A, A))$. Note that in $D(k)$ the complex $\mathrm{RHom}_{A^e}(A, A)$ is isomorphic to the Hochschild cochain complex or Hochschild cocomplex of A. From a recollement of derived categories of algebras, we can obtain three triangles on Hochschild cocomplexes of these algebras, which can induce three long exact sequences on their Hochschild cohomologies.

Lemma 3. *Let A be an algebra and $X \xrightarrow{u} Y \xrightarrow{v} Z \to$ a triangle in $D(A)$ such that $\mathrm{RHom}_A(X, Z) = 0$ in $D(k)$. Then there are three triangles in $D(k)$:*

$$\begin{array}{ll}
(1) & \mathrm{RHom}_A(Y, X) \to \mathrm{RHom}_A(Y, Y) \xrightarrow{\phi} \mathrm{RHom}_A(Z, Z) \to \\
(2) & \mathrm{RHom}_A(Z, Y) \to \mathrm{RHom}_A(Y, Y) \xrightarrow{\psi} \mathrm{RHom}_A(X, X) \to \\
(3) & \mathrm{RHom}_A(Z, X) \to \mathrm{RHom}_A(Y, Y) \xrightarrow{\varphi} \mathrm{RHom}_A(X, X) \oplus \mathrm{RHom}_A(Z, Z) \to .
\end{array}$$

Moreover, ϕ (resp. ψ, φ) induces a homomorphism of graded rings $\bar{\phi}$ (resp. $\bar{\psi}$, $\bar{\varphi}$) between the corresponding cohomology rings.

Proof. Applying the bifunctor $\mathrm{RHom}_A(-, -)$ to the triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \to$, we have the following commutative diagram:

$$\begin{array}{ccccccc}
\mathrm{RHom}_A(X[1], Z[-1]) & \to & \mathrm{RHom}_A(X[1], X) & \to & \mathrm{RHom}_A(X[1], Y) & \to & \mathrm{RHom}_A(X[1], Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{RHom}_A(Z, Z[-1]) & \to & \mathrm{RHom}_A(Z, X) & \to & \mathrm{RHom}_A(Z, Y) & \to & \mathrm{RHom}_A(Z, Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{RHom}_A(Y, Z[-1]) & \to & \mathrm{RHom}_A(Y, X) & \to & \mathrm{RHom}_A(Y, Y) & \to & \mathrm{RHom}_A(Y, Z) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{RHom}_A(X, Z[-1]) & \to & \mathrm{RHom}_A(X, X) & \to & \mathrm{RHom}_A(X, Y) & \to & \mathrm{RHom}_A(X, Z)
\end{array}$$

in which the four corners are zero by the assumption $\mathrm{RHom}_A(X, Z) = 0$ in $D(k)$. It follows two triangles (1) and (2).

By Octahedral axiom, we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & \mathrm{RHom}_A(Z, Z[-1]) & = & \mathrm{RHom}_A(Z, Z[-1]) & & \\
 & & \downarrow & & \downarrow & & \\
\mathrm{RHom}_A(Z, X) & \to & \mathrm{RHom}_A(Y, X) & \to & \mathrm{RHom}_A(X, X) & \to & \mathrm{RHom}_A(Z[-1], X) \\
\| & & \downarrow & & \downarrow & & \| \\
\mathrm{RHom}_A(Z, X) & \to & \mathrm{RHom}_A(Y, Y) & \to & \mathrm{RHom}_A(X, X) \oplus \mathrm{RHom}_A(Z, Z) & \to & \mathrm{RHom}_A(Z[-1], X) \\
 & & \downarrow & & \downarrow & & \\
 & & \mathrm{RHom}_A(Z, Z) & = & \mathrm{RHom}_A(Z, Z) & &
\end{array}$$

where the morphism $\mathrm{RHom}_A(Z, Z[-1]) \to \mathrm{RHom}_A(X, X)$ is zero. It follows the triangle (3).

For the last statement, it is enough to note that ϕ induces a map

$$\bar{\phi} : \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{D(A)}(Y, Y[n]) \to \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{D(A)}(Z, Z[n])$$

sending $f_n \in \mathrm{Hom}_{D(A)}(Y, Y[n])$ to $\bar{\phi}(f_n) \in \mathrm{Hom}_{D(A)}(Z, Z[n])$ such that $\bar{\phi}(f_n) \circ v = v[n] \circ f_n$, i.e., the following diagram in $D(A)$ is commutative:

$$\begin{array}{rcl} Y & \xrightarrow{v} & Z \\ f_n \downarrow & & \downarrow \bar{\phi}(f_n) \\ Y[n] & \xrightarrow{v[n]} & Z[n], \end{array}$$

which is clearly a homomorphism of graded rings. Similar for $\bar{\psi}$ and $\bar{\varphi}$. $\square$

Let A and B be algebras. Denote by $\mathrm{rep}(B, A)$ the full subcategory of $D(B^{\mathrm{op}} \otimes A)$ consisting of all complexes of B-A-bimodules which are perfect when restricted to complexes of A-modules.

Lemma 4. *Let A and B be algebras. Then the derived functor* $\mathrm{RHom}_A(-, A) : D(B^{\mathrm{op}} \otimes A) \to D(A^{\mathrm{op}} \otimes B)$ *induces a dual from* $\mathrm{rep}(B, A)$ *to* $\mathrm{rep}(B^{\mathrm{op}}, A^{\mathrm{op}})$.

Proof. The functor $\mathrm{RHom}_A(-, A)$ (resp. $\mathrm{RHom}_{A^{\mathrm{op}}}(-, A)$) is a contravariant functor from $D(B^{\mathrm{op}} \otimes A)$ to $D(A^{\mathrm{op}} \otimes B)$ (resp. from $D(A^{\mathrm{op}} \otimes B)$ to $D(B^{\mathrm{op}} \otimes A)$). Let I be a homotopically injective resolution of A as A-A-bimodule. Then the functors $\mathrm{RHom}_A(-, A)$ (resp. $\mathrm{RHom}_{A^{\mathrm{op}}}(-, A)$) and $\mathrm{Hom}_A(-, I)$ (resp. $\mathrm{Hom}_{A^{\mathrm{op}}}(-, I)$) are natural isomorphisms.

If $X \in \mathrm{rep}(B, A)$ then, in $D(A^{\mathrm{op}})$, $\mathrm{RHom}_A(X, A) \cong \mathrm{Hom}_A(X_A, I) \cong \mathrm{RHom}_A(X_A, A)$ is perfect. Thus the functor $\mathrm{RHom}_A(-, A)$ can be restricted to a functor from $\mathrm{rep}(B, A)$ to $\mathrm{rep}(B^{\mathrm{op}}, A^{\mathrm{op}})$. Analogously, $\mathrm{RHom}_{A^{\mathrm{op}}}(-, A)$ can be restricted to a functor from $\mathrm{rep}(B^{\mathrm{op}}, A^{\mathrm{op}})$ to $\mathrm{rep}(B, A)$.

We have clearly a natural transformation $\phi : 1 \to \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(-, I), I)$ between these two functors from $\mathrm{rep}(B, A)$ to $\mathrm{rep}(B, A)$. Now it is enough to show that ϕ_X is an isomorphism in $\mathrm{rep}(B, A)$ or $D(B^{\mathrm{op}} \otimes A)$, i.e., ϕ_X is a quasi-isomorphism of complexes of B-A-bimodules. If we can prove that ϕ_X is a quasi-isomorphism of complexes of A-modules, we will be done. This is clear, since $\mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X, I), I)_A = \mathrm{Hom}_{A^{\mathrm{op}}}(\mathrm{Hom}_A(X_A, I), I) \cong \mathrm{RHom}_{A^{\mathrm{op}}}(\mathrm{RHom}_A(X_A, A), A) \cong X_A$ in $D(A)$. $\square$

The main result in this section is the following:

Theorem 13. *Let A_1, A and A_2 be algebras, and $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ a standard recollement given by $Y \in D(A^{\mathrm{op}} \otimes A_1)$ and $Y_2 \in D(A_2^{\mathrm{op}} \otimes A)$. Then there are three triangles in $D(k)$:*

$$\begin{array}{ll}
(1) & \mathrm{RHom}_{A^e}(A, \mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2) \\
 & \to \mathrm{RHom}_{A^e}(A, A) \xrightarrow{\phi} \mathrm{RHom}_{A_1^e}(A_1, A_1) \to \\
(2) & \mathrm{RHom}_{A^e}(\mathrm{RHom}_{A_1}(Y, Y), A) \\
 & \to \mathrm{RHom}_{A^e}(A, A) \xrightarrow{\psi} \mathrm{RHom}_{A_2^e}(A_2, A_2) \to \\
(3) & \mathrm{RHom}_{A^e}(\mathrm{RHom}_{A_1}(Y, Y), \mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2) \\
 & \to \mathrm{RHom}_{A^e}(A, A) \xrightarrow{\varphi} \mathrm{RHom}_{A_1^e}(A_1, A_1) \oplus \mathrm{RHom}_{A_2^e}(A_2, A_2) \to .
\end{array}$$

Moreover, ϕ (resp. ψ, φ) induces a homomorphism of graded rings $\bar{\phi}$ (resp. $\bar{\psi}$, $\bar{\varphi}$) between Hochschild cohomology rings.

Proof. By Theorem 10, we have a recollement $(D(A^{\mathrm{op}} \otimes A_1), D(A^{\mathrm{op}} \otimes A), D(A^{\mathrm{op}} \otimes A_2), I^*, I_* = I_!, I^!, J_!, J^! = J^*, J_*)$ such that

$$\begin{array}{ll}
I^* \cong - \otimes^L_A Y, & J_! \cong - \otimes^L_{A_2} Y_2, \\
I_* = I_! \cong \mathrm{RHom}_{A_1}(Y, -), & J^! = J^* \cong \mathrm{RHom}_A(Y_2, -), \\
I^! \cong \mathrm{RHom}_A(\mathrm{RHom}_{A_1}(Y, A_1), -), & J_* \cong \mathrm{RHom}_{A_2}(\mathrm{RHom}_A(Y_2, A), -).
\end{array}$$

Thus we obtain a triangle $J_!J^!A \to A \to I_*I^*A \to$ in $D(A^{\mathrm{op}} \otimes A)$. By Lemma 3, we have three triangles in $D(k)$:

$$\begin{array}{ll}
(1) & \mathrm{RHom}_{A^e}(A, J_!J^!A) \to \mathrm{RHom}_{A^e}(A, A) \to \mathrm{RHom}_{A^e}(I_*I^*A, I_*I^*A) \to \\
(2) & \mathrm{RHom}_{A^e}(I_*I^*A, A) \to \mathrm{RHom}_{A^e}(A, A) \to \mathrm{RHom}_{A^e}(J_!J^!A, J_!J^!A) \to \\
(3) & \mathrm{RHom}_{A^e}(I_*I^*A, J_!J^!A) \to \mathrm{RHom}_{A^e}(A, A) \to \mathrm{RHom}_{A^e}(J_!J^!A, J_!J^!A) \oplus \\
 & \mathrm{RHom}_{A^e}(I_*I^*A, I_*I^*A) \to .
\end{array}$$

By Lemma 4 and Theorem 10, we have

$$\begin{array}{rcl}
\mathrm{RHom}_{A^e}(J_!J^!A, J_!J^!A) & \cong & \mathrm{RHom}_{A^{\mathrm{op}} \otimes A_2}(J^!A, J^!A) \\
 & \cong & \mathrm{RHom}_{A^{\mathrm{op}} \otimes A_2}(\mathrm{RHom}_A(Y_2, A), \mathrm{RHom}_A(Y_2, A)) \\
 & \cong & \mathrm{RHom}_{A_2^{\mathrm{op}} \otimes A}(Y_2, Y_2) \\
 & \cong & \mathrm{RHom}_{A_2^e}(A_2, A_2)
\end{array}$$

and

$$\begin{array}{rcl}
\mathrm{RHom}_{A^e}(I_*I^*A, I_*I^*A) & \cong & \mathrm{RHom}_{A^{\mathrm{op}} \otimes A_1}(I^*A, I^*A) \\
 & \cong & \mathrm{RHom}_{A^{\mathrm{op}} \otimes A_1}(Y, Y) \\
 & \cong & \mathrm{RHom}_{A_1^{\mathrm{op}} \otimes A}(\mathrm{RHom}_{A_1}(Y, A_1), \mathrm{RHom}_{A_1}(Y, A_1)) \\
 & \cong & \mathrm{RHom}_{A_1^e}(A_1, A_1).
\end{array}$$

Thus there are three triangles in $D(k)$ as required.

For the last statement, it is enough to note that ϕ induces a map

$$\bar{\phi} : \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{D(A^e)}(A, A[n]) \to \oplus_{n\in\mathbb{Z}}\mathrm{Hom}_{D(A_1^e)}(A_1, A_1[n])$$

sending $f_n \in \mathrm{Hom}_{D(A^e)}(A, A[n])$ to $\bar{\phi}(f_n) \in \mathrm{Hom}_{D(A_1^e)}(A_1, A_1[n])$ such that the following diagram in $D(A_1^{\mathrm{op}} \otimes A)$ is commutative:

$$\begin{array}{ccc}
\mathrm{RHom}_{A_1}(Y, A_1) & \overset{\mathrm{RHom}_{A_1}(Y,\bar{\phi}(f_n))}{\longrightarrow} & \mathrm{RHom}_{A_1}(Y, A_1[n]) \\
\downarrow\cong & & \| \\
\mathrm{RHom}_{A_1}(Y[n], A_1[n]) & \overset{\mathrm{RHom}_{A_1}(f_n\otimes^L_A Y, A_1[n])}{\longrightarrow} & \mathrm{RHom}_{A_1}(Y, A_1[n]),
\end{array}$$

which is clearly a homomorphism of graded rings. Similar for $\bar{\psi}$ and $\bar{\varphi}$. $\square$

From the three triangles in Theorem 13, by taking cohomologies, we can obtain three long exact sequences on the Hochschild cohomologies of the algebras:

Corollary 3. *Let A_1, A and A_2 be algebras, and $(D(A_1), D(A), D(A_2), i^*, i_* = i_!, i^!, j_!, j^! = j^*, j_*)$ a standard recollement given by $Y \in D(A^{\mathrm{op}} \otimes A_1)$ and $Y_2 \in D(A_2^{\mathrm{op}} \otimes A)$. Then there are three long exact sequences:*

$$\begin{array}{lll}
(1) & \cdots \to & \mathrm{Ext}^n_{A^e}(A, \mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2) \\
 & \to & HH^n(A) \overset{\phi_n}{\to} HH^n(A_1) \to \cdots \\
(2) & \cdots \to & \mathrm{Ext}^n_{A^e}(\mathrm{RHom}_{A_1}(Y, Y), A) \\
 & \to & HH^n(A) \overset{\psi_n}{\to} HH^n(A_2) \to \cdots \\
(3) & \cdots \to & \mathrm{Ext}^n_{A^e}(\mathrm{RHom}_{A_1}(Y, Y), \mathrm{RHom}_A(Y_2, A) \otimes^L_{A_2} Y_2) \\
 & \to & HH^n(A) \overset{\varphi_n}{\to} HH^n(A_1) \oplus HH^n(A_2) \to \cdots .
\end{array}$$

Moreover, $\oplus_{n\in\mathbb{N}}\phi_n$ (resp. $\oplus_{n\in\mathbb{N}}\psi_n$, $\oplus_{n\in\mathbb{N}}\varphi_n$) is a homomorphisms of graded rings between Hochschild cohomology rings.

Applying Corollary 3 to Example 1, we can obtain the following result due to König and Nagase:

Corollary 4. (König-Nagase [25]) *Let A be an algebra, e an idempotent of A and AeA a stratifying ideal of A. Then there are three long exact sequences:*

$$\begin{array}{ll}
(1) & \cdots \to \mathrm{Ext}^n_{A^e}(A, eAe) \to HH^n(A) \overset{\psi_n}{\to} HH^n(A/AeA) \to \cdots \\
(2) & \cdots \to \mathrm{Ext}^n_{A^e}(A/AeA, A) \to HH^n(A) \overset{\phi_n}{\to} HH^n(eAe) \to \cdots \\
(3) & \cdots \to \mathrm{Ext}^n_{A^e}(A/AeA, eAe) \to HH^n(A) \overset{\varphi_n}{\to} HH^n(A/AeA) \oplus HH^n(eAe) \to \cdots .
\end{array}$$

Moreover, $\oplus_{n\in\mathbb{N}}\phi_n$ (resp. $\oplus_{n\in\mathbb{N}}\psi_n$, $\oplus_{n\in\mathbb{N}}\varphi_n$) is a homomorphisms of graded rings between Hochschild cohomology rings.

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